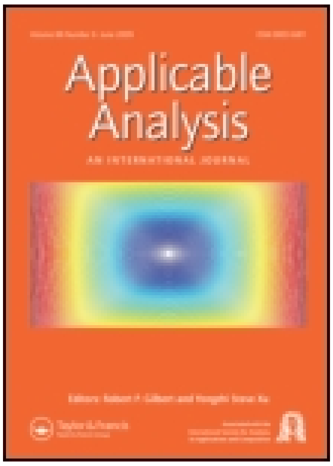

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Loop algebra and visaoro symmetries of integrable hierarchies of KP type

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Loop-Algebra and Virasoro Symmetries of Integrable Hierarchies of KP Type

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We propose a systematic treatment of symmetries of KP integrable systems, including constrained (reduced) KP models $cKP_{R,M}$ (generalized AKNS hierarchies), and their multi-component (matrix) generalizations. Any $cKP_{R,M}$ integrable hierarchy is shown to possess$\left(\widehat{U}(1) \oplus \widehat{SL}(M)\right)_+ \oplus \left(\widehat{SL}(M+R)\right)_-$ loop-algebra (additional) symmetry. Also we provide a systematic construction of the full algebra of Virasoro additional symmetries in the case of constrained KP models which requires a non-trivial modification of the known Orlov–Schulman construction for the general unconstrained KP hierarchy. Multi-component KP hierarchies are identified as ordinary (scalar) one-component KP hierarchies supplemented with the Cartan subalgebra of the additional symmetry algebra, which provides the basis of a new method for construction of soliton-like solutions. Davey–Stewartson and N-wave resonant systems arise as symmetry flows of ordinary $cKP_{R,M}$ hierarchies.

Keywords: KP hierarchy; Loop algebra; Virasoro algebra; Symmetry flows

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1 INTRODUCTION

The Kadomtsev–Petviashvili (KP) hierarchy of integrable soliton evolution equations, together with its reductions and multi-component (matrix) generalizations, describe a variety of physically important non-linear phenomena (for a review, see e.g. [1,2]). Constrained (reduced) KP models are intimately connected with the matrix models in non-perturbative string theory of elementary particles at ultra-high energies ([3] and references therein). They provide an unified description of a number of basic soliton equations such as Korteveg-de-Vries, non-linear Schrödinger (AKNS hierarchy in general), Yajima-Oikawa, coupled Boussinesq-type equations etc. Recently, it was found in [4] that dispersionless KP models play a fundamental role in the description of interface dynamics (Laplacian growth problem). Multi-component (matrix) KP hierarchies, in turn, contain such physically interesting systems as 2-dimensional Toda lattice, Davey–Stewartson, N-wave resonant system etc. Recently it has been shown in [5] that multi-component KP tau-functions provide solutions to the basic Witten–Dijkgraaf–Verlinde–Verlinde equations in topological field theory.

In the present paper we propose a systematic approach, within Sato pseudo-differential operator framework, for treating symmetries of KP integrable systems, including constrained KP models $cKP_{R,M}$ (generalized AKNS hierarchies – see Eq. (20) below), and their multi-component generalizations. Any $cKP_{R,M}$ hierarchy is shown to possess $(\widehat{U}(1) \oplus \widehat{SL}(M))_+ \oplus \left(\widehat{SL}(M+R)\right)_-$ loop-algebra (additional) symmetry generated by squared eigenfunction potentials. The latter subscripts (±) indicate taking the positive or negative-grade part of the corresponding loop algebra. The symmetry flows generating the above two mutually commuting loop-algebra factors will be called "positive"/"negative" for brevity.

Furthermore, we provide a systematic construction of the full algebra of Virasoro additional symmetries in the case of constrained KP models which requires a non-trivial modification of the known Orlov–Schulman Virasoro construction in [6] for the general unconstrained KP hierarchy.

Multi-component (matrix) KP hierarchies are identified as ordinary (scalar) one-component KP hierarchies supplemented with a special set of commuting additional symmetries, namely, the Cartan subalgebra of the underlying loop algebra. This identification leads to new systematic methods of constructing soliton-like solutions of multi-component KP hierachies by employing the well-established techniques of transformations in ordinary one-component KP hierarchies. In particular, Davey–Stewartson [7] and N-wave resonant systems arise as symmetry flows of ordinary $cKP_{R,M}$ hierarchies.

2 SATO FORMALISM FOR ADDITIONAL SYMMETRIES OF INTEGRABLE HIERARCHIES

The general one-component (scalar) KP hierarchy is given by a pseudo-differential[1] Lax operator $\mathcal{L}$ obeying Sato evolution equations (also known as isospectral flow equations; for a systematic exposition, see [2])

$$\mathcal{L} = D + \sum_{k=1}^{\infty} u_k D^{-k}, \quad \frac{\partial}{\partial t_n}\mathcal{L} = \left[(\mathcal{L}^n)_+, \mathcal{L}\right] \tag{1}$$

with Sato dressing operator W

$$\mathcal{L} = WDW^{-1}, \quad \frac{\partial}{\partial t_n} W = -\left(WD^n W^{-1}\right)_- W, \quad W = \sum_{k=0}^{\infty} \frac{p_k(-[\partial])\tau(t)}{\tau(t)} D^{-k} \tag{2}$$

and (adjoint) Baker-Akhiezer (BA) wave functions $\psi_{BA}^{(*)}(t, \lambda)$

$$\mathcal{L}^{(*)}\psi_{BA}^{(*)} = \lambda \psi_{BA}^{(*)}, \quad \frac{\partial}{\partial t_n}\psi_{BA}^{(*)} = \pm(\mathcal{L}^{(*)}n)_+(\psi_{BA}^{(*)}) \tag{3}$$

[1]In what follows the operator D is such that $[D, f] = \partial f = \partial f/\partial x$ and the generalized Leibniz rule holds: $D^n f = \sum_{j=0}^{\infty}\binom{n}{j}(\partial^j f)D^{n-j}$ with $n \in \mathbb{Z}$. In order to avoid confusion we shall employ the following notations: for any (pseudo-)\differential operator $A = \sum_k a_k D^k$ and a function f, the symbol $A(f)$ will indicate application (action) of A on f, whereas the symbol Af will denote simply operator product of A with the zero-order (multiplication) operator f. Projections ($\pm$) are defined as: $A_+ = \sum_{k\geq 0} a_k D^k$ and $A_- = \sum_{k\leq -1} a_k D^k$. Finally, $A \equiv a_{-1}$.

$$\psi^{(*)}_{BA}(t,\lambda) = W^{(*-1)}\left(e^{\pm\xi(t,\lambda)}\right) = \frac{\tau\left(t \mp [\lambda^{-1}]\right)}{\tau(t)}\, e^{\pm\xi(t,\lambda)}, \quad \xi(t,\lambda) \equiv \sum_{\ell=1}^{\infty} t_\ell \lambda^\ell \tag{4}$$

where the tau-function $\tau(t)$ satisfies the relation:

$$\partial_x \frac{\partial}{\partial t_n} \ln \tau = \mathrm{Res}\mathcal{L}^n \tag{5}$$

Here and below we employ the following short-hand notations: $(t) \equiv (t_1 \equiv x, t_2, \ldots)$ for the set of isospectraltime-evolution parameters; $[\partial] \equiv \left(\partial/\partial t_1,\, \frac{1}{2}\partial/\partial t_2,\, \frac{1}{3}\partial/\partial t_3, \ldots\right)$ and $[\lambda^{-1}] \equiv \left(\lambda^{-1},\, \frac{1}{2}\lambda^{-2},\, \frac{1}{3}\lambda^{-3}, \ldots\right)$; $p_k(.)$ indicate the well-known Schur polynomials.

There exist few other objects in Sato formalism for integrable hierarchies which play fundamental role in our construction. (Adjoint) eigenfunctions $\Phi(t)$ ($\Psi(t)$, respectively) are those functions of KP "times" (t) satisfying:

$$\frac{\partial}{\partial t_l}\Phi = (\mathcal{L}^l)_+(\Phi), \quad \frac{\partial}{\partial t_l}\Psi = -(\mathcal{L}^l)^*_+(\Psi) \tag{6}$$

According to second Eq. (3), (adjoint) BA functions are special cases of (adjoint) eigenfunctions, which in addition satisfy spectral equations (first Eq. (3)).

It has been shown in [8] that any (adjoint) eigenfunction possesses a spectral representation of the form[2]:

$$\Phi(t) = \int d\lambda\, \varphi(\lambda)\, \psi_{BA}(t,\lambda), \quad \Psi(t) = \int d\lambda\, \psi(\lambda)\, \psi^*_{BA}(t,\lambda) \tag{7}$$

with appropriate spectral densities $\varphi(\lambda)$ and $\psi(\lambda)$ which are formal Laurent series in λ. Clearly, any KP hierarchy possesses an infinite set of independent (adjoint) eigenfunctions in one-to-one correspondence with the space of all independent formal Laurent series in λ.

[2]Integrals over spectral parameters are understood as: $\int d\lambda \equiv \oint_0 d\lambda/2i\pi = \mathrm{Res}\lambda = 0$.

The next important object is the so called squared eigenfunction potential (SEP) [9] – a function $S(\Phi(t), \Psi(t))$ associated with an arbitrary pair of (adjoint) eigenfunctions $\Phi(t), \Psi(t)$ which possesses the following characteristics:

$$\frac{\partial}{\partial t_n} S(\Phi(t), \Psi(t)) = \mathrm{Res}\big(D^{-1}\Psi(\mathcal{L}^n)_+ \Phi D^{-1}\big) \tag{8}$$

In particular, for $n = 1$ Eq. (8) implies $\partial_x S(\Phi(t), \Psi(t)) = \Phi(t)\,\Psi(t)$ (recall $\partial_x \equiv \partial/\partial t_1$). Equation (8) determines $S(\Phi(t), \Psi(t)) \equiv \partial^{-1}(\Phi(t)\,\Psi(t))$ up to a shift by a trivial constant which is uniquely fixed by the fact that any SEP obeys the following double-spectral representation [8]:

$$\begin{aligned}\partial^{-1}(\Phi(t)\,\Psi(t)) &= -\int\!\!\int d\lambda\, d\mu\, \psi(\lambda)\, \varphi(\mu)\, \frac{1}{\lambda}\, \psi^*_{BA}(t, \lambda)\psi_{BA}\big(t + [\lambda^{-1}], \mu\big)\\ &= -\int\!\!\int d\lambda\, d\mu\, \frac{\psi(\lambda)\,\varphi(\mu)}{\lambda - \mu}\, e^{\xi(t,\mu)-\xi(t,\lambda)}\, \frac{\tau\big(t + [\lambda^{-1}] - [\mu^{-1}]\big)}{\tau(t)}\end{aligned}$$

with $\varphi(\lambda)$, $\psi(\lambda)$ being the respective spectral densities in (7). It is in this well-defined sense that inverse space derivatives ∂^{-1} will appear throughout our construction below.

A flow on the space of Sato pseudo-differential Lax operators $\mathcal{L}$ or, equivalently, on the space of Sato dressing operators W is given by:

$$\delta_\alpha \mathcal{L} = [\mathcal{M}_\alpha, \mathcal{L}], \quad \delta_\alpha W = \mathcal{M}_\alpha W \tag{10}$$

where $\mathcal{M}_\alpha$ is a purely pseudo-differential operator. A flow δ_α (10) is a symmetry if and only if it commutes with the isospectral flows $\partial/\partial t_l$:

$$\left[\delta_\alpha, \frac{\partial}{\partial t_l}\right] = 0 \quad \longrightarrow \quad \frac{\partial}{\partial t_l}\mathcal{M}_\alpha = \big[(\mathcal{L}^l)_+, \mathcal{M}_\alpha\big]_- \tag{11}$$

The general form of $\mathcal{M}_\alpha$ obeying (11) is provided by [10]:

$$\mathcal{M}_\alpha = \int\!\!\int d\lambda d\mu\, \rho_\alpha(\lambda, \mu)\, \psi_{BA}(t, \mu) D^{-1} \psi^*_{BA}(t, \lambda) = \sum_{I,J\in\mathcal{A}} c^{(\alpha)}_{IJ}\, \Phi_J D^{-1} \Psi_I \tag{12}$$

where $\rho_\alpha(\lambda, \mu)$ is arbitrary (in the case of the general unconstrained KP hierarchy) double Laurent series in λ and μ. In the second equality above the sums run in general over an infinite set $\mathcal{A}$ of indices, and $\{\Phi_I, \Psi_I\}_{I\in\mathcal{A}}$ are (adjoint) eigenfunctions of the Lax operator $\mathcal{L}$ (6) for $\Phi = \Phi_I$ and $\Psi = \Psi_I$. The second equality in (12) arises from the general representation of the "bispectral" density:

$$\rho_\alpha(\lambda, \mu) = \sum_{I,J\in\mathcal{A}} c_{IJ}^{(\alpha)} \varphi_J(\mu)\psi_I(\lambda) \tag{13}$$

in terms of basis of Laurent series $\{\varphi_I(\mu)\}$ and $\{\psi_I(\lambda)\}$, together with spectral representation theorem in [8] for (adjoint) eigenfunctions of Sato Lax operators (cf. Eq. (7)).

Our main objective below will be to construct explicitly symmetry-flow generating operators (12), such that the corresponding flows both preserve the constrained form of Lax operators defining constrained KP hierarchies, as well as they yield a closed algebra of symmetries. At this point let us recall that the following special form of $\rho_\alpha(\lambda, \mu) = \lambda^k(\partial/\partial\lambda)^l\delta(\lambda - \mu)$ in (12) [10] yields the well-known Orlov–Schulman $W_{1+\infty}$ additional symmetries [6] in the case of the general unconstrained KP hierarchy. On the other hand, these standard Orlov–Schulman symmetry flows fail to produce symmetries in the case of constrained KP hierarchies since they do not preserve the constrained form of the pertinent Lax operators. The solution to this problem is provided by a non-trivial modification of the former flows (see [11] and Section 6 below).

On any (adjoint) eigenfunction the action of the flow δ_α (10) takes the form:

$$\delta_\alpha\Phi = \mathcal{M}_\alpha(\Phi) + \mathcal{F}^{(\alpha)}, \quad \delta_\alpha\Psi = -\mathcal{M}_\alpha^*(\Psi) + \mathcal{G}^{(\alpha)} \tag{14}$$

where $\mathcal{F}^{(\alpha)}$ and $\mathcal{G}^{(\alpha)}$ are other (adjoint) eigenfunctions. Equations (14) follow from the second Eq. (10) taking again into account the spectral representation theorem in [8]. Note that the emergence of the additional (adjoint) eigenfunctions terms on the r.h.s. of (14) is due to the fact that the spectral densities of Φ andΨ in (7) may in general vary under the action of δ_α. Moreover, as it will be seen in Section 4 below, in the case of constrained KP hierarchies the presence of the additional terms on the r.h.s. of Eqs. (14) is mandatory for consistency of the flow action (10) with the constrained form of the pertinent Lax operator, which accordingly uniquely fixes the form of $\mathcal{F}^{(\alpha)}$ and $\mathcal{G}^{(\alpha)}$.

Making use of the well-known pseudo-differential operator identities (cf. e.g. the appendix in first ref. [11]) :

$$M_1 M_2 = M_1(f_2)D^{-1}g_2 + f_1 D^{-1} M_2^*(g_1)$$
$$M_{1,2} \equiv f_{1,2}D^{-1}g_{1,2}, \quad M_1(f_2) = f_1\partial^{-1}(g_1 f_2) \text{ etc.}$$

one easily finds that the flows δ_a (10) span (an infinite-dimensional, in general) closed algebra:

$$\left[\delta_\alpha, \delta_\beta\right]\mathcal{L} = \left[\delta_\alpha \mathcal{M}_\beta - \delta_\beta \mathcal{M}_\alpha - [\mathcal{M}_\alpha, \mathcal{M}_\beta], \mathcal{L}\right] \tag{16}$$

where

$$\delta_\alpha \mathcal{M}_\beta - \delta_\beta \mathcal{M}_\alpha - [\mathcal{M}_\alpha, \mathcal{M}_\beta] \equiv \mathcal{M}_{[\alpha,\beta]}$$
$$= \sum_{I,J\in A} \left(c^{(\beta)}a^{(\alpha)} - c^{(\alpha)}a^{(\beta)} + b^{(\alpha)}c^{(\beta)} - b^{(\beta)}c^{(\alpha)}\right)_{IJ} \Phi_J D^{-1} \Psi_I \tag{17}$$

Here the matrices $a_{IJ}^{(\alpha)}$ and $b_{IJ}^{(\alpha)}$ appear in the inhomogeneous terms in the δ_α-flow equations for Φ_I and Ψ_I, respectively, according to Eq. (14):

$$\delta_\alpha \Phi_I = \mathcal{M}_\alpha(\Phi_I) + \sum_J a_{IJ}^{(\alpha)} \Phi_J, \quad \delta_\alpha \Psi_I = -\mathcal{M}_\alpha^*(\Psi_I) + \sum_J b_{JI}^{(\alpha)} \Psi_J \tag{18}$$

and similarly for $a_{IJ}^{(\beta)}$ and $b_{IJ}^{(\beta)}$. In the case of general KP hierarchy the specific form of $a_{IJ}^{(\alpha)}$ and $b_{IJ}^{(\alpha)}$ for any flow (with label α) is arbitrary *a priori*, and it is subject to the only condition of fulfillment of Jacobi identities for the flow commutator (17). However, for constrained KP hierarchies the form of $a_{IJ}^{(\alpha)}$ and $b_{IJ}^{(\alpha)}$ is determined uniquely from the consistency (Eq. (21) below) of the flow action with the constrained form of the pertinent Lax operator, see Section 4 below.

Finally, starting from relation (5) and using (11) we find for the transformation of the tau-function under the action of δ_α-flow (10):

$$\delta_\alpha \ln \tau = -\partial^{-1}(\mathrm{Res}\mathcal{M}_\alpha) = -\sum_{I,J} c_{IJ}^{(\alpha)} \partial^{-1}(\Phi_J \Psi_I) \tag{19}$$

3 CONSTRAINED KP HIERARCHIES. INVERSE POWERS OF LAX OPERATORS

So far we have considered the general case of unconstrained KP hierarchy. Now we are interested in symmetries for *constrained* KP hierarchies $\mathsf{cKP}_{R,M}$ with Lax operators (cf. [11,8,12] and references therein):

$$\mathcal{L} \equiv \mathcal{L}_{R,M} = D^R + \sum_{i=0}^{R-2} v_i D^i + \sum_{j=1}^{M} \Phi_j D^{-1} \Psi_j = L_{M+R} L_M^{-1} \tag{20}$$

where $\{\Phi_i, \Psi_i\}_{i=1}^M$ is a set of (adjoint) eigenfunctions of $\mathcal{L}$.

The second representation of $\mathcal{L} \equiv \mathcal{L}_{R,M}$[3] is in terms of a ratio of two monic purely differential operators L_{M+R} and L_M of orders $M+R$ and M, respectively (see [12] and references therein). For $\mathcal{L} \equiv \mathcal{L}_{R,M}$ the Sato evolution (isospectral flow) Eqs. (1), the equations for (adjoint) BA (3) and (adjoint) eigenfunctions (6) acquire the form:

$$\frac{\partial}{\partial t_n}\mathcal{L} = \left[\left(\mathcal{L}^{n/R}\right)_+, \mathcal{L}\right], \quad \mathcal{L}^{(*)}\psi_{BA}^{(*)} = \lambda^R \psi_{BA}^{(*)}, \quad \frac{\partial}{\partial t_n}\psi_{BA}^{(*)} = \pm\left(\mathcal{L}^{(*)}\right)_+^{n/R}\left(\psi_{BA}^{(*)}\right) \tag{21}$$

$$\frac{\partial}{\partial t_n}\Phi = \left(\mathcal{L}^{n/R}\right)_+(\Phi), \quad \frac{\partial}{\partial t_n}\Psi = -\left(\mathcal{L}^{n/R}\right)_+^*(\Psi) \tag{22}$$

In the case of constrained hierarchies (20), we have the following additional condition on the symmetry generating operator $\mathcal{M}_\alpha$ since the flow (10) must preserve the constrained form (20) of the pertinent Lax operator (cf. (14) and 12)):

$$\sum_{i=1}^{M}\left[\mathcal{F}_i^{(\alpha)} D^{-1}\Psi_i + \Phi_i D^{-1}\mathcal{G}_i^{(\alpha)}\right] = \sum_{I,J\in\{\alpha\}} c_{IJ}\left[\Phi_J^{(\alpha)} D^{-1}\mathcal{L}^*\left(\Psi_I^{(\alpha)}\right) - \mathcal{L}\left(\Phi_J^{(\alpha)}\right) D^{-1}\Psi_I^{(\alpha)}\right] \tag{23}$$

[3]Henceforth we shall employ the short-hand notation $\mathcal{L}$ for $\mathcal{L}_{R,M}$ (20) whenever this will not lead to a confusion.

where (cf. (14)):

$$\delta_\alpha \Phi_i = \mathcal{M}_\alpha(\Phi_i) + \mathcal{F}_i^{(\alpha)}, \quad \delta_\alpha \Psi_i = -\mathcal{M}_\alpha^*(\Psi_i) + \mathcal{G}_i^{(\alpha)} \tag{24}$$

Equation (21) uniquely fixes the form of the additional terms $\mathcal{F}_i^{(\alpha)}$ and $\mathcal{G}_i^{(\alpha)}$ in (22).

In what follows we will also need the δ_α-flow equations on inverse powers of the Lax operator $\mathcal{L} = L_{M+R} L_M^{-1}$ (20). First, let us recall that the inverses of the underlying purely differential operators are given by

$$L_M^{-1} = \sum_{i=1}^{M} \varphi_i D^{-1} \psi_i, \quad L_{M+R}^{-1} = \sum_{a=1}^{M+R} \bar{\varphi}_a D^{-1} \bar{\psi}_a \tag{25}$$

where the functions $\{\varphi_i\}_{i=1}^M$ and $\{\psi_i\}_{i=1}^M$ span $Ker(L_M)$ and $Ker(L_M^*)$, respectively, whereas $\{\bar{\varphi}_a\}_{a=1}^{M+R}$ and $\{\bar{\psi}_a\}_{a=1}^{M+R}$ span $Ker(L_{M+R})$ and $Ker(L_{M+R}^*)$, respectively. Therefore we have

$$\mathcal{L} = (\mathcal{L})_+ + \sum_{i=1}^{M} L_{M+R}(\varphi_i) D^{-1} \psi_i, \quad \text{i.e. } \Phi_i = L_{M+R}(\varphi_i)\ , \quad \Psi_i = \psi_i \tag{26}$$

$$\mathcal{L}^{-1} = \sum_{a=1}^{M+R} L_M(\bar{\varphi}_a) D^{-1} \bar{\psi}_a \tag{27}$$

$$\mathcal{L}^{-N} = \sum_{a=1}^{M+R} \sum_{s=0}^{N-1} \mathcal{L}^{-(N-1)+s}\big(L_M(\bar{\varphi}_a)\big) D^{-1} \big(\mathcal{L}^{-s}\big)^*(\bar{\psi}_a) \tag{28}$$

Compare the last formula (26) with the formula in [13] for the negative pseudo-differential part of a positive power of $\mathcal{L}$ (20)

$$\big(\mathcal{L}^N\big)_- = \sum_{i=1}^{M} \sum_{s=0}^{N-1} \mathcal{L}^{N-1-s}(\Phi_i) D^{-1} \big(\mathcal{L}^s\big)^*(\Psi_i) \tag{29}$$

Let us also note that the following simple consequences from the definitions of the corresponding objects will play essential role for the consistency of the constructions involving inverse powers of $\mathcal{L}$:

$$\mathcal{L}\big(L_M(\bar{\varphi}_a)\big) = 0, \quad \mathcal{L}^*(\bar{\psi}_a) = 0, \quad \mathcal{L}^{-1}(\Phi_i) = 0, \quad \big(\mathcal{L}^{-1}\big)^*(\Psi_i) = 0 \tag{30}$$

Applying the flow Eq. (10) to $\mathcal{L}^{-1}$ (25) $\delta_\alpha \mathcal{L}^{-1} = [\mathcal{M}_\alpha, \mathcal{L}^{-1}]$ and taking into account the explicit form of $\mathcal{M}_\alpha$ (second equality 12)) we obtain

$$\delta_\alpha\big(L_M(\bar{\varphi}_a)\big) = \mathcal{M}_\alpha\big(L_M(\bar{\varphi}_a)\big) + \bar{\mathcal{F}}_a^{(\alpha)}, \quad \delta_\alpha \bar{\psi}_a = -\mathcal{M}_\alpha^*(\bar{\psi}_a) + \bar{\mathcal{G}}_a^{(\alpha)} \tag{31}$$

with consistency condition for the "shift" functions $\bar{\mathcal{F}}_a^{(\alpha)}$and $\bar{\mathcal{G}}_a^{(\alpha)}$ (the analog of Eq. (22))

$$\sum_{a=1}^{M+R}\Big[\bar{\mathcal{F}}_a^{(\alpha)} D^{-1}\bar{\psi}_a + L_M(\bar{\varphi}_a) D^{-1}\bar{\mathcal{G}}_a^{(\alpha)}\Big]$$
$$= \sum_{I,J\in\mathcal{A}} c_{IJ}^{(\alpha)}\Big[\Phi_J D^{-1}\big(\mathcal{L}^{-1}\big)^*(\Psi_I) - \mathcal{L}^{-1}(\Phi_J)D^{-1}\Psi_I\Big] \tag{32}$$

Also, from the isospectral flows equations applied on $\mathcal{L}^{-1}$, i.e., $\partial(\mathcal{L}^{-1})/\partial t_n = [\mathcal{L}_+^{n/R}, \mathcal{L}^{-1}]$, we find, taking into account (28), that $L_M(\bar{\varphi}_a)$ and $\bar{\psi}_a$ are (adjoint) eigenfunctions of $\mathcal{L}$ (cf. (??)):

$$\frac{\partial}{\partial t_n} L_M(\bar{\varphi}_a) = \big(\mathcal{L}^{n/R}\big)_+\big(L_M(\bar{\varphi}_a)\big), \quad \frac{\partial}{\partial t_n}\bar{\psi}_a = -\big(\mathcal{L}^{n/R}\big)_+^*(\bar{\psi}_a) \tag{33}$$

4 LOOP-ALGEBRA SYMMETRIES OF KP HIERARCHIES

Let us consider the following system of M infinite sets of (adjoint) eigenfunctions of $\mathcal{L} \equiv \mathcal{L}_{R,M}$ (20):

$$\Phi_i^{(n)} \equiv \mathcal{L}^{n-1}(\Phi_i), \quad \Psi_i^{(n)} \equiv (\mathcal{L}^*)^{n-1}(\Psi_i), \quad n = 1, 2, \ldots; \quad i = 1, \ldots, M \tag{34}$$

which are expressed in terms of the M pairs of (adjoint) eigenfunctions entering the pseudo-differential part of $\mathcal{L} \equiv \mathcal{L}_{R,M}$ (20). Using (32) we can build the following infinite set of symmetry flows (cf. (10) and 12)):

$$\delta_A^{(n)}\mathcal{L} = \Big[\mathcal{M}_A^{(n)}, \mathcal{L}\Big], \quad \mathcal{M}_A^{(n)} \equiv \sum_{i,j=1}^{M} A_{ij}^{(n)} \sum_{s=1}^{n} \Phi_j^{(n+1-s)} D^{-1}\Psi_i^{(s)} \tag{35}$$

where $A^{(n)}$ is an arbitrary constant $M \times M$ matrix, i.e., $A^{(n)} \in Mat(M)$. Consistency of the flow action (33) with the constrained form (20) of

$\mathcal{L} \equiv \mathcal{L}_{R,M}$ (cf. (21)) implies the following flow action on the involved (adjoint) eigenfunctions

$$\delta_A^{(n)}\Phi_i^{(m)} = \mathcal{M}_A^{(n)}(\Phi_i^{(m)}) - \sum_{j=1}^{M} A_{ij}^{(n)}\Phi_j^{(n+m)}$$

$$\delta_A^{(n)}\Psi_i^{(m)} = -\left(\mathcal{M}_A^{(n)}\right)^*(\Psi_i^{(m)}) + \sum_{j=1}^{M} A_{ji}^{(n)}\Psi_j^{(n+m)} \qquad (36)$$

The specific form of the inhomogeneous terms on the r.h.s. of Eqs. (34) is the main ingredient of our construction. It is precisely these inhomogeneous terms which yield non-trivial loop-algebra additional symmetries.

Using the pseudo-differential operator identities (15) and taking into account (34) we can show that (cf. Eq. (17))

$$\delta_A^{(n)}\mathcal{M}_B^{(m)} - \delta_B^{(m)}\mathcal{M}_A^{(n)} - \left[\mathcal{M}_A^{(n)}, \mathcal{M}_B^{(m)}\right] = \mathcal{M}_{[A,\,B]}^{(n+m)} \qquad (37)$$

Equation (35) implies that the symmetry flows (33)–(34) span the following infinite-dimensional algebra

$$\left[\delta_A^{(n)}, \delta_B^{(m)}\right] = \delta_{[A,\,B]}^{(n+m)}; \quad A^{(n)}, B^{(m)} \in \mathrm{Mat}(M) \ , \ n,m = 1,2,\ldots \qquad (38)$$

isomorphic to $(\widehat{U}(1) \times \widehat{SL}(M))_+$ where the subscript (+) indicates taking the positive-grade sub algebra of the corresponding loop-algebra. We observe, that in the case of $\mathsf{cKP}_{R,M}$ models we have $\mathcal{M}_{A=\mathbb{1}}^{(n)} = (\mathcal{L}_{R,M}^n)_-$ (insert (32) into first relation (33) for $A^{(n)} = \mathbf{1}$ and compare with (27)). Therefore, the flows $\delta_{A=\mathbb{1}}^{(n)}$ for $\mathsf{cKP}_{R,M}$ models coincide upto a sign with the ordinary isospectral flows modulo R: $\delta_{A=\mathbb{1}}^{(n)} = -\partial/\partial t_{nR}$ (cf. Eq. (??)). Thereby the flows $\delta_A^{(n)}$ (33) will be called "positive" for brevity.

Now we consider another infinite set of (adjoint) eigenfunctions of $\mathcal{L} \equiv \mathcal{L}_{R,M}$ expressed in terms of the (adjoint) eigenfunctions entering the inverse power of $\mathcal{L} \equiv \mathcal{L}_{R,M}$ (25) :

$$\Phi_a^{(-m)} \equiv \mathcal{L}^{-(m-1)}\big(L_M(\bar{\varphi}_a)\big), \quad \Psi_a^{(-m)} \equiv \left(\mathcal{L}^{-(m-1)}\right)^*(\bar{\psi}_a) \qquad (39)$$

$$m = 1, 2, \ldots, \quad a = 1, \ldots, M + R \tag{40}$$

Using (37) we obtain the following set of "negative" symmetry flows which parallels completely the set of "positive" flows (33):

$$\delta_{\mathcal{A}}^{(-n)} \mathcal{L} = \left[\mathcal{M}_{\mathcal{A}}^{(-n)}, \mathcal{L} \right], \quad \mathcal{M}_{\mathcal{A}}^{(-n)} \equiv \sum_{a,b=1}^{M+R} \mathcal{A}_{ab}^{(-n)} \sum_{s=1}^{n} \Phi_b^{(-n-1+s)} D^{-1} \Psi_a^{(-s)} \tag{41}$$

where $\mathcal{A}_{ab}^{(-n)}$ is an arbitrary constant $(M+R) \times (M+R)$ matrix, i.e., $\mathcal{A}^{(-n)} \in \mathrm{Mat}(M+R)$. In fact, since according to (26) we have $\mathcal{M}_{\mathcal{A}=\mathbb{1}}^{(-n)} = \mathcal{L}^{-n}$, the flows $\delta_{\mathcal{A}=\mathbb{1}}^{(-n)}$ vanish identically, i.e., $\delta_{\mathcal{A}=\mathbb{1}}^{(-n)} = 0$, therefore, we restrict $\mathcal{A}^{(-n)} \in SL(M+R)$.

Consistency of the flow action (39) with the constrained form (20) of $\mathcal{L} \equiv \mathcal{L}_{R,M}$ (cf. (21)) and with the constrained form (25) of the inverse $\mathcal{L}^{-1}$ implies the following $\delta_{\mathcal{A}}^{(-n)}$-flow action on the involved (adjoint) eigenfunctions (using short-hand notations (32) and (37)):

$$\delta_{\mathcal{A}}^{(-n)} \Phi_i^{(m)} = \mathcal{M}_{\mathcal{A}}^{(-n)} (\Phi_i^{(m)}), \quad \delta_{\mathcal{A}}^{(-n)} \Psi_i^{(m)} = -\left(\mathcal{M}_{\mathcal{A}}^{(-n)} \right)^* (\Psi_i^{(m)}) \tag{42}$$

$$\delta_{\mathcal{A}}^{(-n)} \Phi_a^{(-m)} = \mathcal{M}_{\mathcal{A}}^{(-n)} (\Phi_a^{(-m)}) - \sum_{b=1}^{M+R} \mathcal{A}_{ab}^{(-n)} \Phi_b^{(-n-m)}$$

$$\delta_{\mathcal{A}}^{(-n)} \Psi_a^{(-m)} = -\left(\mathcal{M}_{\mathcal{A}}^{(-n)} \right)^* (\Psi_a^{(-m)}) + \sum_{b=1}^{M+R} \mathcal{A}_{ba}^{(-n)} \Psi_b^{(-n-m)} \tag{43}$$

Similarly, consistency of "positive" $\delta_{\mathcal{A}}^{(n)}$-flow action (33) with the constrained form (25) of the inverse Lax operator implies:

$$\delta_{\mathcal{A}}^{(n)} \Phi_a^{(-m)} = \mathcal{M}_{\mathcal{A}}^{(n)} (\Phi_a^{(-m)}), \quad \delta_{\mathcal{A}}^{(n)} \Psi_a^{(-m)} = -\left(\mathcal{M}_{\mathcal{A}}^{(n)} \right)^* (\Psi_a^{(-m)}) \tag{44}$$

Using again the pseudo-differential operator identities (15) we find from (40)–(42) (cf. Eq. (35))

$$\delta_{\mathcal{A}}^{(n)} \mathcal{M}_{\mathcal{B}}^{(-m)} - \delta_{\mathcal{B}}^{(-m)} \mathcal{M}_{\mathcal{A}}^{(n)} - \left[\mathcal{M}_{\mathcal{A}}^{(n)}, \mathcal{M}_{\mathcal{B}}^{(-m)} \right] = 0 \tag{45}$$

$$\delta_{\mathcal{A}}^{(-n)} \mathcal{M}_{\mathcal{B}}^{(-m)} - \delta_{\mathcal{B}}^{(-m)} \mathcal{M}_{\mathcal{A}}^{(-n)} - \left[\mathcal{M}_{\mathcal{A}}^{(-n)}, \mathcal{M}_{\mathcal{B}}^{(-m)} \right] = \mathcal{M}_{[\mathcal{A},\mathcal{B}]}^{(-n-m)} \tag{46}$$

Equations (43)–(44) imply that the "negative" symmetry flows (20)–(41) commute with the "positive" flows (33)–(34)

$$\left[\delta_{\mathcal{A}}^{(n)}, \delta_{\mathcal{B}}^{(-m)}\right] = 0 \tag{47}$$

and that they themselves span the following infinite-dimensional algebra

$$\left[\delta_{\mathcal{A}}^{(-n)}, \delta_{\mathcal{B}}^{(-m)}\right] = \delta_{[\mathcal{A},\mathcal{B}]}^{(-n-m)}; \quad \mathcal{A}^{(-n)}, \mathcal{B}^{(-m)} \in SL(M+R), \;\; n,m = 1,2,\dots \tag{48}$$

which is isomorphic to $(\widehat{SL}(M+R))_-$ (the subscript $(-)$ indicates taking the negative-grade subalgebra of the corresponding loop-algebra).

Therefore, we conclude that the full loop algebra of (additional) symmetries of $\mathsf{cKP}_{R,M}$ hierarchies (20) is the direct sum

$$\left(\widehat{U}(1) \oplus \widehat{SL}(M)\right)_+ \oplus \left(\widehat{SL}(M+R)\right)_- \tag{49}$$

The construction above can be straightforwardly extended to the case of the general unconstrained KP hierarchy defined by (1). All relations (33)–(36) and (39)–(46) remain intact where now

$$\left\{\Phi_i^{(n)}, \Psi_i^{(n)}\right\}_{i=1,\dots,M}^{n=1,2,\dots}, \quad \left\{\Phi_a^{(-n)}, \Psi_a^{(-n)}\right\}_{a=1,\dots,M+R}^{n=1,2,\dots} \tag{50}$$

form an infinite system of independent (adjoint) eigenfunctions of the general Lax operator (1) with M, $M+R$ being arbitrary positive integers.

5 MULTI-COMPONENT KP HIERARCHIES FROM ONE-COMPONENT ONES

Let us now consider the following subset of "positive" flows $\delta_{E_k}^{(n)}$ (33) for the general KP hierarchy (1) corresponding to

$$E_k = \mathrm{diag}(0,\dots 0,1,0,\dots,0), \quad \text{i.e.} \quad \mathcal{M}_{E_k}^{(n)} = \sum_{s=1}^{n} \Phi_k^{(n+1-s)} D^{-1} \Psi_k^{(s)} \tag{51}$$

Due to Eq. (36) the flows $\delta^{(n)}_{E_k}$ span an infinite-dimensional Abelian algebra and, by construction, they commute with the original isospectral flows $\partial/\partial t_n$ as well. Comparison with our construction in [14] allows us to identify the set of isospectral flows plus the set of $\delta^{(n)}_{E_k}$-flows (49)

$$\frac{\partial}{\partial t_n} \equiv \partial/\partial \overset{(1)}{t_n}, \quad \delta^{(n)}_{E_k} \equiv \partial/\partial \overset{(k+1)}{t_n}, \quad k = 1, \ldots, M \tag{52}$$

with the set of isospectral flows $\{\overset{(\ell)}{t_n}\}^{\ell=1,\ldots,M+1}_{n=1,2,\ldots}$ of the (unconstrained) $M+1$-component matrix KP hierarchy. The latter is defined in terms of the $M+1 \times M+1$ matrix Hirota bilinear identities (see [14])

$$\sum_{k=1}^{M+1} \varepsilon_{ik}\,\varepsilon_{jk} \int d\lambda\, \lambda^{\delta_{ik}+\delta_{jk}-2}\, e^{\xi(\overset{(k)}{t}-\overset{(k)}{t'},\lambda)} \tau_{ik}\big(\ldots, \overset{(k)}{t} - [\lambda^{-1}], \ldots\big) \\ \times \tau_{kj}\big(\ldots, \overset{(k)}{t'} + [\lambda^{-1}], \ldots\big) = 0 \tag{53}$$

which are obeyed by a set of $M(M+1)+1$ tau-functions $\{\tau_{ij}\}$ expressed in terms of the single tau-function τ and the "positive" symmetry flow generating (adjoint) eigenfunctions (48) in the original one-component (scalar) KP hierarchy (1)–(5) as follows

$$\tau_{11} = \tau_{ii} = \tau, \quad \tau_{1i} = \tau\,\Phi^{(1)}_{i-1}, \quad \tau_{i1} = -\tau\,\Psi^{(1)}_{i-1} \tag{54}$$

$$\tau_{ij} = \varepsilon_{ij}\tau\,\partial^{-1}\Big(\Phi^{(1)}_{j-1}\Psi^{(1)}_{i-1}\Big), \quad i \neq j \ , \ i,j = 2, \ldots M+1 \tag{55}$$

Here $\varepsilon_{ij} = 1$ for $i \leq j$ and $\varepsilon_{ij} = -1$ for $i > j$, and δ_{ij} are the usual Kronecker symbols.

The above construction of multi-component (matrix) KP hierarchies out of ordinary one-component ones can be straightforwardly carried over to the case of constrained KP models (20) using the identification (32) for the symmetry-generating (adjoint) eigenfunctions. In this case, however, there is a linear dependence among the flows (50) $\sum_{k=1}^{M} \delta^{(n)}_{E_k} = -\partial/\partial t_n$, therefore, the associated constrained multi-component KP hierarchy is now $M \times M$ matrix hierarchy.

Similarly, we can start with the subset of "negative" symmetry flows $\delta^{(-n)}_{E_k}$ (20) for $cKP_{R,M}$ hierarchy[4]

$$\delta^{(-n)}_{E_k} \equiv \partial/\partial \overset{(k)}{t}_{-n}, \quad \mathcal{M}^{(-n)}_{E_k} = \sum_{s=1}^{n} \Phi^{(-n-1+s)}_{k} D^{-1} \Psi^{(-s)}_{k},$$
$$k = 2, \ldots, M+R, \quad n = 1, 2, \ldots \tag{56}$$

which also span an infinite-dimensional Abelian algebra of flows commuting with the isospectral flows. Then, following the steps of our construction in [14] we arrive at $(M+R)$-component constrained KP hierarchy given in terms of $(M+R)(M+R-1)+1$ tau-functions $\{\widetilde{\tau}_{ab}\}$ obeying the corresponding $(M+R) \times (M+R)$ matrix Hirota bilinear identities (cf. (51)). The latter tau-functions are expressed in terms of the original single tau-function τ and the "negative" flow generating (adjoint) eigenfunctions (37) in the original ordinary $cKP_{R,M}$ hierarchy as follows:

$$\widetilde{\tau}_{11} = \widetilde{\tau}_{aa} = \tau, \quad \widetilde{\tau}_{1a} = \tau\, L_M(\bar{\varphi}_a), \quad \widetilde{\tau}_{a1} = -\tau\, \bar{\psi}_a \tag{57}$$

$$\widetilde{\tau}_{ab} = \varepsilon_{ab} \tau\, \partial^{-1}\big(L_M(\bar{\varphi}_b)\bar{\psi}_a\big), \quad a \neq b, \quad a, b = 2, \ldots M+R \tag{58}$$

Let us recall that multi-component (matrix) KP hierarchies (51) contain various physically interesting non-linear systems such as Davey–Stewartson and N-wave systems, which now can be written entirely in terms of objects belonging to ordinary one-component (constrained) KP hierarchy. For instance, the N-wave resonant system ($N = M(M+1)/2$) is given by

$$\partial_c f_{ij} = f_{ik} f_{kj}, \quad i \neq j \neq k, \quad i, j, k = 1, \ldots, M+1 \tag{59}$$

$$\partial_k \equiv \partial/\partial \overset{(k)}{t}_1, \quad f_{1i} \equiv \Phi^{(1)}_{i-1}, \quad f_{i1} \equiv -\Psi^{(1)}_{i-1} \tag{60}$$

$$f_{ij} \equiv \varepsilon_{ij} \partial^{-1}\Big(\Phi^{(1)}_{j-1} \Psi^{(1)}_{i-1}\Big), \quad i \neq j, \quad i, j = 2, \ldots M+1 \tag{61}$$

[4]The flow $\delta^{(-n)}_{E_k}$ for $k = 1$ is excluded since $\sum_{k=1}^{M+R} \delta^{(-n)}_{E_k} = \delta^{(-n)}_{A=1}$ which vanishes identially as explained in the previous section.

As a further example, we will demonstrate that the well-known Davey–Stewartson system in [7] arises as particular subset of symmetry flow equations obeyed by any pair of adjoint eigenfunctions (Φ_i, Ψ_i) (i = fixed) or $\big(L_M(\bar{\varphi}_a), \bar{\psi}_a\big)$ (a = fixed). In fact, for (Φ_i, Ψ_i) (i = fixed) this has already been done in [14]. Here for simplicity we take $cKP_{1,M}$ hierarchy (Eq. (20) with $R = 1$; the general case is straight-forward generalization of the formulas below) and consider a pair of "negative" symmetry flow generating (adjoint) eigenfunctions, e.g., $(\phi \equiv L_M(\bar{\varphi}_a), \psi \equiv \bar{\psi}_a)$ (a = fixed), which obeys the following subset of flow equations – w.r.t. $\partial/\partial t_2$, $\bar{\partial} \equiv \partial/\partial \overset{(a)}{t}_{-1}$ and $\partial/\partial\bar{t}_2 \equiv \partial/\partial \overset{(a)}{t}_{-2}$ (cf. Eq. (41)):

$$\frac{\partial}{\partial t_2}\phi = \left(\partial^2 + 2\sum_{i=1}^{M}\Phi_i\Psi_i\right)\phi, \quad \frac{\partial}{\partial t_2}\psi = -\left(\partial^2 + 2\sum_{i=1}^{M}\Phi_i\Psi_i\right)\psi \tag{62}$$

$$\bar{\partial}\phi = \mathcal{M}_1^{(-1)}(\phi) - \mathcal{L}^{-1}(\phi), \quad \bar{\partial}\psi = -\left(\mathcal{M}_1^{(-1)}\right)^*(\psi) + \left(\mathcal{L}^{-1}\right)^*(\psi) \tag{63}$$

$$\partial/\partial\bar{t}_2\phi = \mathcal{M}_1^{(-2)}(\phi) - \mathcal{L}^{-2}(\phi), \quad \partial/\partial\bar{t}_2\psi = -\left(\mathcal{M}_1^{(-2)}\right)^*(\psi) + \left(\mathcal{L}^{-2}\right)^*(\psi) \tag{64}$$

where

$$\mathcal{M}_1^{(-1)} \equiv \phi D^{-1}\psi, \quad \mathcal{M}_1^{(-2)} \equiv \mathcal{L}^{-1}(\phi)D^{-1}\psi + \phi D^{-1}\left(\mathcal{L}^{-1}\right)^*(\psi) \tag{65}$$

Using (61) we can rewrite Eq. (62) as purely differential equation w.r.t. $\bar{\partial}$

$$\frac{\partial}{\partial\bar{t}_2}\phi = \left[-\bar{\partial}^2 + 2\bar{\partial}\big(\partial^{-1}(\phi\psi)\big)\right]\phi, \quad \frac{\partial}{\partial\bar{t}_2}\psi = \left[\bar{\partial}^2 - 2\bar{\partial}\big(\partial^{-1}(\phi\psi)\big)\right]\psi \tag{66}$$

Now, introducing new time variable $T = t_2 - \bar{t}_2$ and the short-hand notation $Q \equiv \sum_{i=1}^{M}\Phi_i\Psi_i - 2(\phi\psi) - 2\bar{\partial}\big(\partial^{-1}(\phi\psi)\big)$, and subtracting Eqs. (64) from Eqs. (60), we arrive at the following system of (2 + 1)-dimensional non-linear evolution equations:

$$\frac{\partial}{\partial T}\phi = \left[\frac{1}{2}(\partial^2 + \bar{\partial}^2) + Q + 2\phi\psi\right]\phi \tag{67}$$

$$\frac{\partial}{\partial T}\psi = -\left[\frac{1}{2}(\partial^2 + \bar{\partial}^2) + Q + 2\phi\psi\right]\psi \tag{68}$$

$$\partial\bar{\partial}Q + (\partial + \bar{\partial})^2(\phi\psi) = 0 \tag{69}$$

which is precisely the standard Davey–Stewartson system in [7] for the "negative" (adjoint) eigenfunction pair $\left(\phi \equiv L_M(\bar{\varphi}_a),\ \psi \equiv \bar{\psi}_a\right)$ ($a =$ fixed).

The construction in this Section allows us to employ the well-known techniques from ordinary one-component (scalar) KP hierarchies (full or constrained) in order to obtain new soliton-like solutions of multi-component (matrix) KP hierarchies (see [14]).

6 THE FULL VIRASORO ALGEBRA OF ADDITIONAL SYMMETRIES

In [11] we have constructed an essential modification to the original Orlov–Schulman additional Virasoro symmetry flows in [6] needed in the case of $\mathsf{cKP}_{R,M}$ reduced KP models (20) for $n \geq 0$, i.e., for the Borel subalgebra (henceforth $\mathcal{L} \equiv \mathcal{L}_{R,M}$)

$$\delta_n^V \mathcal{L} = \left[-(M\mathcal{L}^n)_- + \mathcal{X}_n, \mathcal{L}\right] \tag{70}$$

or, equivalently

$$\delta_n^V \mathcal{L} = \left[(M\mathcal{L}^n)_+ + \mathcal{X}_n, \mathcal{L}\right] + \mathcal{L}^n \tag{71}$$

where $\delta_n^V \simeq -L_{n-1}$ (in terms of standard Virasoro notations). Here

$$[\mathcal{L}, M] = \mathbf{1}, \quad M = \sum_{k\geq 1} k t_k \mathcal{L}^{k-1} + \sum_{j\geq 1}\left(-j p_j(-[\partial]) \ln \tau\right)\mathcal{L}^{-j-1} \tag{72}$$

$$\mathcal{X}_n \equiv \sum_{i=1}^{M}\sum_{j=0}^{n-2}\left(j - \tfrac{1}{2}(n-2)\right)\mathcal{L}^{n-2-j}(\Phi_i)D^{-1}\left(\mathcal{L}^j\right)^*(\Psi_i) \tag{73}$$

The presence of the additional terms $\mathcal{X}_n$ in (68) is very crucial to ensure that the flows δ_n^V preserve the constrained form of the pertinent pseudo-differential Lax operator (20). The ordinary Orlov–Schulman flows $\delta_n^{OS}\mathcal{L} = \left[-(M\mathcal{L}^n)_-, \mathcal{L}\right]$ do not define symmetries for constrained $\mathsf{cKP}_{R,M}$ hierarchies.

The action of δ_n^V-flows on the pertinent (adjoint) eigenfunctions reads accordingly (for $n \geq 0$) :

$$\delta_n^V \Phi_i = \left[(M\mathcal{L}^n)_+ + \mathcal{X}_n\right](\Phi_i) + \frac{n}{2}\mathcal{L}^{n-1}(\Phi_i) \tag{74}$$

$$\delta_n^V \Psi_i = -\left[(M\mathcal{L}^n)_+^* + \mathcal{X}_n^*\right](\Psi_i) + \frac{n}{2}\left(\mathcal{L}^{n-1}\right)^*(\Psi_i) \tag{75}$$

Similarly, for the (adjoint) eigenfunctions entering the inverse Lax powers we find from $\delta_n^V \mathcal{L}^{-1} = \left[-(M\mathcal{L}^n)_- + \mathcal{X}_n, \mathcal{L}^{-1}\right]$ and Eqs. (28) (for $n \geq 0$):

$$\delta_n^V L_M(\bar{\varphi}_a) = \left[(M\mathcal{L}^n)_+ + \mathcal{X}_n\right]\left(L_M(\bar{\varphi}_a)\right), \quad \delta_n^V \bar{\psi}_a = -\left[(M\mathcal{L}^n)_+^* + \mathcal{X}_n^*\right](\bar{\psi}_a) \tag{76}$$

Here we want to extend the above construction to cover the case of the full Virasoro algebra of additional symmetries. For the negative flows we must therefore find the appropriate additional terms $\mathcal{X}_{(-n)}$

$$\delta_{-n}^V \mathcal{L} = \left[-(M\mathcal{L}^{-n})_- + \mathcal{X}_{(-n)}, \mathcal{L}\right] \tag{77}$$

or, equivalently

$$\delta_{-n}^V \mathcal{L} = \left[(M\mathcal{L}^{-n})_+ + \mathcal{X}_{(-n)}, \mathcal{L}\right] + \mathcal{L}^{-n} \tag{78}$$

so that the consistency condition (21) is satisfied. Using again the pseudo-differential operator identities (15) and taking into account the relevant formulas for negative Lax powers (26) we obtain the following explicit expressions for $\mathcal{X}_{(-n)}$

$$\mathcal{X}_{(-n)} = \sum_{a=1}^{M+R}\sum_{j=0}^{n}\left(\frac{n}{2} - j\right)\mathcal{L}^{-(n-j)}\left(L_M(\bar{\varphi}_a)\right)D^{-1}\left(\mathcal{L}^{-j}\right)^*(\bar{\psi}_a) \tag{79}$$

The consistency of the negative flow definitions (76) with $\mathcal{X}_{(-n)}$ as in Eq. (77) crucially depends on the relations (28).

The flows δ^V_{-n} act on the constituent (adjoint) eigenfunctions of $\mathcal{L}$ as

$$\delta^V_{-n}\Phi_i = \left[(M\mathcal{L}^{-n})_+ + \mathcal{X}_{(-n)}\right](\Phi_i), \quad \delta^V_{-n}\Psi_i = -\left[(M\mathcal{L}^{-n})^*_+ + \mathcal{X}^*_{(-n)}\right](\Psi_i) \tag{80}$$

and similarly on the (adjoint) eigenfunctions $L_M(\bar{\varphi}_a)$, $\bar{\psi}_a$ entering the inverse powers of $\mathcal{L}$

$$\delta^V_{-n}\big(L_M(\bar{\varphi}_a)\big) = \left[(M\mathcal{L}^{-n})_+ + \mathcal{X}_{(-n)}\right]\big(L_M(\bar{\varphi}_a)\big) - \Big(\frac{n}{2}+1\Big)\mathcal{L}^{-(n+1)}\big(L_M(\bar{\varphi}_a)\big) \tag{81}$$

$$\delta^V_{-n}\bar{\psi}_a = -\left[(M\mathcal{L}^{-n})^*_+ + \mathcal{X}^*_{(-n)}\right](\bar{\psi}_a) - \Big(\frac{n}{2}+1\Big)\big(\mathcal{L}^{-(n+1)}\big)^*(\bar{\psi}_a) \tag{82}$$

Let us now consider the commutator of the Virasoro flows $\delta^V_n \simeq -L_{n-1}$ and $\delta^V_m \simeq -L_{m-1}$ acting on $\mathcal{L}$ (cf. Eq. (16)) where (n,m) are arbitrary non-negative or negative indices

$$\begin{aligned}\left[\delta^V_n, \delta^V_m\right] = \delta^V_n\big(-(M\mathcal{L}^m)_- + \mathcal{X}_m\big) - \delta^V_m\big(-(M\mathcal{L}^n)_- + \mathcal{X}_n\big) \\ - \left[-(M\mathcal{L}^n)_- + \mathcal{X}_n,\ -(M\mathcal{L}^m)_- + \mathcal{X}_m\right]\end{aligned} \tag{83}$$

Using the identity

$$\begin{aligned}\delta^V_n(M\mathcal{L}^m)_- - \delta^V_m(M\mathcal{L}^n)_- = -(n-m)\big(M\mathcal{L}^{n+m-1}\big)_- - \left[(M\mathcal{L}^n)_-, (M\mathcal{L}^m)_-\right] \\ + [\mathcal{X}_n, M\mathcal{L}^m]_- - [\mathcal{X}_m, M\mathcal{L}^n]_-\end{aligned} \tag{84}$$

the r.h.s. of Eq. (81) can be rewritten in the form

$$\begin{aligned}(n-m)\big(M\mathcal{L}^{n+m-1}\big)_- + \delta^V_n\mathcal{X}_m - \left[(M\mathcal{L}^n)_+, \mathcal{X}_m\right]_- \\ - \delta^V_m\mathcal{X}_n + \left[(M\mathcal{L}^m)_+, \mathcal{X}_n\right]_- - [\mathcal{X}_n, \mathcal{X}_m]\end{aligned} \tag{85}$$

Now, employing the pseudo-differential identities (15) it is easy to show, taking into account (72)–(74) and (78)–(80), that the sum of all terms in (83) involving $\mathcal{X}_{n,m}$ yield

$$\begin{aligned}&\delta^V_n\mathcal{X}_m - \left[(M\mathcal{L}^n)_+, \mathcal{X}_m\right]_- - \delta^V_m\mathcal{X}_n + \left[(M\mathcal{L}^m)_+, \mathcal{X}_n\right]_- - [\mathcal{X}_n, \mathcal{X}_m] \\ &\quad = -(n-m)\mathcal{X}_{n+m-1}\end{aligned} \tag{86}$$

Thus, we have verified the closure of the full Virasoro algebra of additional symmetries without central extension

$$[\delta_n^V, \delta_m^V] = -(n-m)\delta_{n+m-1}^V \tag{87}$$

OUTLOOK

In a subsequent paper we will generalize the present construction of additional symmetries to the case of supersymmetric integrable hierarchies. We will continue the derivation and study of properties of new soliton-like solutions of multi-component KP hierarchies obtained via standard methods for ordinary one-component KP models which has been already initiated in [14]. In a forthcoming more detailed paper we will systematically study the construction of additional (loop-algebra and Virasoro) symmetries within a generalized Drinfeld–Sokolov formalism both in ordinary and supersymmetric integrable systems of KP type. Also we will relate the algebraic dressing method to Sato pseudo-differential operator approach.

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